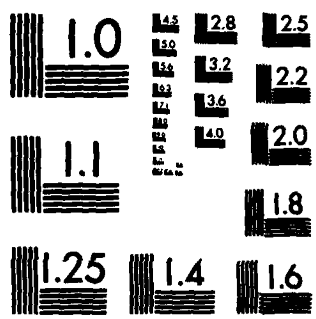
AD-A120 984 SOLITARY AND PERIODIC GRAVITY-CAPILLARY WAVES OF FINITE 1/1
AMPLITUDE(U) WISCONSIN UNIV-MADISON MATHEMATICS
RESEARCH CENTER J K HUNTER ET AL. AUG 82 MRC-TSR-2414
UNCLASSIFIED DAAG29-80-C-0041 F/G 20/4 NL

END
DATE
FILMED
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

MRC Technical Summary Report #2414

SOLITARY AND PERIODIC GRAVITY-CAPILLARY WAVES OF FINITE AMPLITUDE

J. K. Hunter and J.-M. Vanden-Broeck

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

August 1982

(Received June 11, 1982)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

SOLITARY AND PERIODIC GRAVITY-CAPILLARY WAVES OF FINITE AMPLITUDE

J. K. Hunter* and J.-M. Vanden-Broeck**

Technical Summary Report #2414
August 1982

ABSTRACT

Two dimensional solitary and periodic waves in water of finite depth are considered. The waves propagate under the combined influence of gravity and surface tension. The flow, the surface profile, and the phase velocity are functions of the amplitude of the wave and the parameters $\ell = \lambda/H$ and $\tau = T/\rho g H^2$. Here λ is the wavelength, H the depth, T the surface tension, ρ the density and g the gravity. For large values of ℓ and small values of the amplitude, the profile of the wave satisfies the Korteweg de Vries equation approximately. However, for τ close to 1/3 this equation becomes invalid. In the present paper a new equation valid for τ close to 1/3 is obtained. Moreover, a numerical scheme based on an integro-differential equation formulation is derived to solve the problem in the fully nonlinear case. Accurate solutions for periodic and solitary waves are presented. In addition, the limiting configuration for large amplitude solitary waves when $\tau > \frac{1}{2}$ is found analytically. Graphs of the results are included.

AMS (MOS) Subject Classifications: 76B25, 76B15

Key Words: Solitary waves, Periodic waves, Surface tension

Work Unit Number 2 (Physical Mathematics)

---

*Department of Mathematics, Colorado State University, Fort Collins, Colorado 80523.

**Department of Mathematics and Mathematics Research Center, University of Wisconsin-Madison, Madison, Wisconsin 53706.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062, Mod. 1 and No. MCS-8001960.

SIGNIFICANCE AND EXPLANATION

The study of the influence of surface tension on nonlinear water waves has attracted much attention in recent years because of the challenging mathematical difficulties associated with the non-uniqueness of the solutions. Several numerical schemes are now available to compute capillary-gravity waves in water of infinite or moderate depth. (Schwartz and Vanden-Broeck (1979), Chen and Saffman (1980)).

In the present work we present a numerical scheme which enables us to compute waves in the limit as the ratio of the depth versus the wavelength tends to zero. A problem of non-uniqueness is discovered and discussed. Moreover, a new equation analogous to the Korteweg de Vries equation is obtained. In addition, the limiting configuration of steep depression solitary waves is obtained analytically.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

SOLITARY AND PERIODIC GRAVITY-CAPILLARY WAVES OF FINITE AMPLITUDE

J. K. Hunter* and J.-M. Vanden-Broeck**

1. Introduction

Approximate solutions for gravity solitary and cnoidal waves of small amplitude were obtained by Rayleigh (1876), and Korteweg and de Vries (1895). These results were derived systematically by Keller (1948) who calculated a first order perturbation solution in powers of the wave amplitude. This work was extended to second order by Laitone (1960).

More recently, accurate fully nonlinear solutions for gravity solitary waves were obtained by Longuet-Higgins and Fenton (1974), Byatt-Smith and Longuet-Higgins (1976), Witting (1975) and Hunter and Vanden-Broeck (1982). A review of some of these theories can be found in Miles (1980).

Accurate solutions for periodic gravity waves in water of finite depth were obtained by Schwartz (1974), Cokelet (1977), Vanden-Broeck and Schwartz (1979), and Rienecker and Fenton (1981). The effect of surface tension on periodic waves was investigated by Crapper (1957), Harrison (1909), Wilton (1915), Pearson and Fife (1961), Schwartz and Vanden-Broeck (1979) and Chen and Saffman (1980). For a review of these calculations see Schwartz and Fenton (1982).

The effect of surface tension on solitary waves was first considered by Korteweg and de Vries (1895). They discovered that depression solitary waves can exist for sufficiently large values of the surface tension. A systematic perturbation calculation was attempted by Shinbrot (1981). However, his results are partially incorrect because he excluded the

*Department of Mathematics, Colorado State University, Fort Collins, Colorado 80523.

**Department of Mathematics and Mathematics Research Center, University of Wisconsin-Madison, Madison, Wisconsin 53706.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062, Mod. 1 and No. MCS-8001960.

possibility of depression waves. A first order perturbation solution allowing depression waves was derived by Vanden-Broeck and Shen (1982) and Benjamin (1982).

This perturbation calculation is invalid when

$$\tau = \frac{T}{\rho g H^2} \tag{1.1}$$

is close to 1/3. In (1.1) T is the surface tension, ρ the fluid density, g the gravitational acceleration, and H is the undisturbed depth of the fluid.

In this paper we present a perturbation calculation valid near $\tau = 1/3$. In addition, we solve the exact nonlinear equations numerically.

In Section 2 we formulate the nonlinear problem and briefly review some of the classical perturbation calculations. In Sections 3 and 4 we describe the perturbation calculation valid for τ close to 1/3. In Section 5 we reformulate the problem as an integro-differential equation on the free surface, which allows us to calculate solitary and periodic waves of arbitrary amplitude. A numerical scheme for solving the integro-differential equation is presented in Section 6. The numerical method is similar in philosophy if not in details to the scheme derived by Vanden-Broeck and Schwartz (1979). The results of the numerical computation are discussed in Section 7. In addition the limiting configuration for large amplitude solitary waves when $\tau > 1/2$ is found analytically in Section 7.

2. Formulation and Classical Perturbation Solutions

We consider a two dimensional progressive wave in an irrotational incompressible inviscid fluid having a free surface with surface tension acting upon it, and bounded below by a flat horizontal bottom. We take a frame of reference in which the flow is steady, with the X-axis parallel to the bottom, and with the Y-axis a line of symmetry of the wave. The phase velocity C is defined as the average fluid velocity at any horizontal level completely within the fluid.

We introduce a potential function $\Phi(X,Y)$ and a stream function $\Psi(X,Y)$. Let the stream function assume the values zero and $-Q$ on the free surface and on the bottom respectively. The undisturbed fluid depth H is defined by

$$H = \frac{Q}{C} . \tag{2.1}$$

We take the origin of our coordinate system on the undisturbed level of the free surface, so that the bottom is given by $Y = -H$, and we denote the equation of the free surface by $Y = \zeta(X)$.

The exact nonlinear equations for Φ and ζ are

$$\Phi_{XX} + \Phi_{YY} = 0, \qquad -H < Y < \zeta(X) , \tag{2.2}$$

$$\Phi_Y = 0 \quad \text{on} \quad Y = -H , \tag{2.3}$$

$$\Phi_X \zeta_X - \Phi_Y = 0 \quad \text{on} \quad Y = \zeta(X) , \tag{2.4}$$

$$\frac{1}{2}\{\Phi_X^2 + \Phi_Y^2\} + g\zeta - \frac{T}{\rho}\frac{\zeta_{XX}}{(1+\zeta_X^2)^{3/2}} = \frac{1}{2} b^2 \quad \text{on} \quad Y = \zeta(X) . \tag{2.5}$$

Equation (2.2) follows from conservation of mass. Equations (2.3) and (2.4) are the conditions that the bottom and free surface are streamlines, and equation (2.5) is the dynamic free surface condition. The Bernoulli constant b on the right hand side of (2.5) is to be found as part of the solution.

In order to derive asymptotic solutions it is convenient to introduce the following dimensionless variables

$$x = \frac{X}{L} , \quad y = \frac{Y}{H} , \quad \eta = \frac{\zeta}{A}$$
$$\phi = \frac{H\Phi}{LA\sqrt{gH}} , \quad \psi = \frac{H\Psi}{LA\sqrt{gH}} , \quad B = \frac{b}{\sqrt{gH}} . \tag{2.6}$$

In (2.6) L is a length scale of the wave in the X direction and A is a measure of the amplitude. Rewriting (2.2)-(2.5) in terms of the dimensionless variables (2.6) gives

$$\beta\phi_{xx} + \phi_{yy} = 0 \qquad -1 < y < \alpha\eta(x) \tag{2.7}$$

$$\phi_y = 0 \quad \text{on} \quad y = -1 \tag{2.8}$$

$$\alpha\eta_x\phi_x - \frac{1}{\beta}\phi_y = 0 \quad \text{on} \quad y = \alpha\eta(x) \tag{2.9}$$

$$\frac{1}{2}\alpha\phi_x^2 + \frac{1}{2}\frac{\alpha}{\beta}\phi_y^2 + \eta - \beta\tau\frac{\eta_{xx}}{(1+\alpha^2\beta\eta_x^2)^{3/2}} = \frac{B^2}{2\alpha} \quad \text{on} \quad y = \alpha\eta(x) \ . \tag{2.10}$$

In (2.7)-(2.10) α and β are the dimensionless parameters

$$\alpha = \frac{A}{H} , \quad \beta = \frac{H^2}{L^2} . \tag{2.11}$$

Relations (2.7)-(2.10) are the classical water wave equations.

We seek solutions for periodic water waves of wavelength λ, and introduce the dimensionless wavelength

$$\tilde{\ell} = \lambda/L \ . \tag{2.12}$$

The Froude number F is defined by

$$F = \frac{c}{\sqrt{gH}} = \frac{\alpha}{\tilde{\ell}}\int_0^{\tilde{\ell}} \phi_x dx \ . \tag{2.13}$$

Solitary waves are the limit of periodic waves as $\tilde{\ell} \to \infty$. In that case $F = B$.

In order to motivate the considerations presented in the next sections, it is worthwhile reviewing briefly some of the classical perturbation solutions of the system (2.7)-(2.11). Stokes (1847, 1880) derived a perturbation solution for pure gravity waves by assuming α small and β of order one. His results were generalized to include the effect of surface tension by Harrison (1909), Wilton (1915) and Nayfeh (1970). However, these perturbation calculations become invalid as $\beta \to 0$ because the ratio of successive terms is then unbounded.

The shallow water equations are derived by assuming β small and α of order one. These equations do not have travelling wave solutions because the dispersive effects are neglected (Whitham (1974), p. 457). The inclusion of dispersive effects into the shallow

water theory leads to the Korteweg de Vries equation. This equation can be derived by assuming β small and α of order β (Keller (1948), Vanden-Broeck and Shen (1982)). Thus we let

$$\alpha = \beta = \varepsilon \tag{2.14}$$

and expand η, ϕ and B as

$$\begin{aligned} \eta &= \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \cdots \\ \phi &= \frac{Bx}{\varepsilon} + \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots \\ B &= B_0 + \varepsilon B_1 + \varepsilon^2 B_2 + \cdots . \end{aligned} \tag{2.15}$$

Then we use (2.14) and (2.15) in (2.7)-(2.10), and find that

$$B_0 = F_0 = 1 \tag{2.16}$$

$$\phi_0' = -\eta_0 \tag{2.17}$$

$$2F_1\eta_0' - 3\eta_0\eta_0' + (\tau - \frac{1}{3})\eta_0''' = 0 . \tag{2.18}$$

The primes in (2.17) and (2.18) denote derivatives with respect to x. The dispersive effect arises from the last term in (2.18).

Korteweg and de Vries (1895) showed that periodic solutions of (2.18) can be found in closed form. As the wavelength tends to infinity these waves tend to the solitary wave solution

$$\begin{gathered} \eta_0 = a \operatorname{sech}^2\left(\frac{x}{b}\right) \\ a = 2F_1, \qquad b = \left[\frac{4(1-3\tau)}{3a}\right]^{1/2} . \end{gathered} \tag{2.19}$$

When $\tau < 1/3$ these are elevation waves with Froude number greater than one when $\tau > 1/3$ they are depression waves with Froude number less than one.

Equation (2.19) shows that the slope of the wave profile becomes large near $\tau = 1/3$ and the solution ceases to exist altogether when $\tau = 1/3$. Thus the Korteweg de Vries equation (2.18) (denoted in the remaining part of the paper as the KdV equation) is invalid in the neighborhood of $\tau = 1/3$. This is due to the fact that the dispersive effects disappear as τ approaches $1/3$.

3. Perturbation Solution Near $\tau = 1/3$

In this section we shall derive an equation analogous to the KdV equation which is valid in a neighborhood of $\tau = 1/3$. The coefficient of the dispersive term η_0''' in (2.18) vanishes at $\tau = 1/3$ making a travelling wave impossible. To obtain a balance between the dispersive and nonlinear terms near $\tau = 1/3$ we take

$$\alpha = \varepsilon^2 \qquad \beta = \varepsilon \tag{3.1}$$

in (2.7)-(2.10). Then we expand η, ϕ, B and τ as

$$\begin{aligned} \eta &= \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \cdots \\ \phi &= \frac{Bx}{\varepsilon^2} + \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \cdots \\ \tau &= \frac{1}{3} + \varepsilon\tau_1 + \varepsilon^2\tau_2 + \cdots \\ B &= B_0 + \varepsilon B_1 + \varepsilon^2 B_2 + \cdots \\ F &= F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots . \end{aligned} \tag{3.2}$$

From (3.1) and (2.7)-(2.8) ϕ satisfies

$$\begin{aligned} \varepsilon\phi_{xx} + \phi_{yy} &= 0 \\ \phi_y = 0 \quad &\text{on} \quad y = -1 . \end{aligned} \tag{3.3}$$

Then using (3.2) in (3.3) and equating coefficients of powers of ε to zero, we obtain

$$\begin{aligned} \phi_0 &= f(x) \\ \phi_1 &= -\frac{1}{2}(1+y)^2 f'' + g(x) \\ \phi_2 &= \frac{1}{4!}(1+y)^4 f^{(iv)} - \frac{1}{2}(1+y)^2 g'' + h(x) \\ \phi_3 &= -\frac{1}{6!}(1+y)^6 f^{(vi)} + \frac{1}{4!}(1+y)^4 g^{iv} - \frac{1}{2}(1+y)^2 h'' + k(x) . \end{aligned} \tag{3.4}$$

In (3.4) f, g, h and k are arbitrary functions of x and primes denote differentiation by x.

We use (3.1), (3.2) and (3.4) in (2.9)-(2.10), expand in power series about $\varepsilon = 0$ and equate coefficients of ε^0, ε^1 and ε^2. This gives the following equations

$$B_0 f' + \eta_0 = 0 \tag{3.5}$$

$$B_0\eta_0' + f'' = 0 \tag{3.6}$$

$$B_0 g' + \eta_1 = \frac{1}{3}\eta_0'' + \frac{1}{2}B_0 f''' - B_1 f' \tag{3.7}$$

$$B_0\eta_1' + g_0'' = \frac{1}{6}f^{(iv)} - B_1\eta_0' \tag{3.8}$$

$$B_0h' + \eta_2 = + \frac{B_0}{2} g''' - \frac{1}{4!} B_0 f^{(v)} - B_2 f' - \frac{1}{2}(f')^2 + \frac{1}{3}\eta_1'' + \tau_1\eta_0'' + B_1(\frac{f''}{2} - g') \qquad (3.9)$$

$$B_0\eta_2' + h'' = -B_2\eta_0' + \frac{1}{6} g^{(iv)} - \eta_0 f'' - \frac{1}{5!} f^{(vi)} - \eta_0' f_0' - B_1\eta_1' \; . \qquad (3.10)$$

For (3.5) to have a nontrivial solution we must have

$$B_0^2 = 1 \; . \qquad (3.11)$$

We shall consider waves moving to the right and take

$$B_0 = 1 \; . \qquad (3.12)$$

Then (3.5) implies

$$\eta_0 = - f' \; . \qquad (3.13)$$

We differentiate (3.7), add the result to (3.8) and use (3.12)-(3.13). Then it follows for a nontrivial solution that

$$B_1 = 0 \; . \qquad (3.14)$$

Using (3.12)-(3.14) in (3.7) gives

$$\eta_1 = -g' + \frac{1}{6} f''' \; . \qquad (3.15)$$

Next we use (3.12)-(3.15) in (3.9)-(3.10), and after eliminating η_2 and h we obtain the following equation for η_0

$$2B_2\eta_0' - 3\eta_0\eta_0' + \tau_1\eta_0''' - \frac{1}{45}\eta_0^{(v)} = 0 \; . \qquad (3.16)$$

Finally we show that the Bernoulli constant B is the same as the Froude number F to the order considered here. From (2.1), (2.6) and (2.11) we have

$$F = \alpha \int_{-1}^{\alpha\eta} \phi_x dy \; . \qquad (3.17)$$

Then using (3.1), (3.2) and (3.4) in (3.17) we find that

$$F_0 = B_0 = 1, \quad F_1 = B_1 = 0, \quad F_2 = B_2 \; . \qquad (3.18)$$

A similar perturbation calculation using (3.1) can be performed on the time dependent water wave equation. This leads to

$$2(gH)^{-1/2}\zeta_t - 2\mathcal{F}_X - \frac{3}{H}\zeta\zeta_X + H^2(\tau - \frac{1}{3})\zeta_{XXX} - \frac{H^4}{45}\zeta_{XXXXX} = 0 \; . \qquad (3.19)$$

Equation (3.19) reduces to (3.16) for travelling wave solutions $\zeta = \zeta(X + ct)$.

In the next section we seek periodic solutions of (3.16) by a Stokes type expansion.

4. Periodic Waves Near $\tau = 1/3$

We seek a periodic solution of (3.16) in the form of an expansion in powers of the wave amplitude. Therefore we write

$$\eta_0 = a\eta_{01} + a^2\eta_{02} + O(a^3) \ , \tag{4.1}$$

$$F_2 = F_{20} + aF_{21} + a^2F_{22} + O(a^3) \ . \tag{4.2}$$

Here a is a measure of the wave amplitude, the precise meaning of it will become clear later.

Next we substitute (4.1) and (4.2) into (3.16) and (3.18) and collect all terms of like powers of a. Thus we obtain

$$2F_{20}\eta_{01}' + \tau_1\eta_{01}''' - \frac{1}{45}\eta_{01}^{(v)} = 0 \ , \tag{4.3}$$

$$2F_{20}\eta_{02}' + \tau_1\eta_{02}''' - \frac{1}{45}\eta_{02}^{(v)} = 3\eta_{01}\eta_{01}' - 2F_{21}\eta_{01}' \ . \tag{4.4}$$

The solutions of (4.3) and (4.4) are given by

$$\eta_{01} = \cos Kx \tag{4.5}$$

$$\eta_{02} = -3K[16\tau_1K^2 - 8KF_{20} - \frac{64}{45}K^5]^{-1}\cos 2Kx \tag{4.6}$$

$$F_{20} = \frac{\tau_1K^2}{2} + \frac{K^4}{90} \tag{4.7}$$

$$F_{21} = 0 \ . \tag{4.8}$$

Here $K = 2\pi/\tilde{\ell}$ is the wavenumber. Relations (4.1) and (4.5) define a as the amplitude of the fundamental in the Fourier series expansion of $\eta_0(x)$.

The classical dispersion relation for linear water waves is given by (see Whitham (1974), p. 403)

$$F^2 = \frac{\lambda}{2\pi H}\tanh\left(\frac{2\pi H}{\lambda}\right)\left(1 + \tau\frac{4\pi^2H^2}{\lambda^2}\right) \ . \tag{4.9}$$

From (2.11), (2.12) and (3.1) we have $2\pi H/\lambda = \epsilon^{1/2}2\pi/\tilde{\ell}$. Substituting this result and (3.2) into (4.9), and expanding for ϵ small we find

$$F^2 = 1 + 2F_{20}$$

where F_{20} is given by (4.7). Thus the solution of (3.16) overlaps the classical linear solution, as the amplitude tends to zero.

The perturbation solution (4.5)-(4.8) is invalid when

$$F_{20} = 2\tau_1K^2 + \frac{8}{45}K^4 \tag{4.10}$$

because the coefficient of $\cos 2Kx$ in (4.6) is then unbounded. When (4.10) is satisfied, the solution of (4.3) is given by

$$\eta_{01} = \cos Kx + E_2 \cos 2Kx \tag{4.11}$$

$$F_{20} = \frac{\tau_1 K^2}{2} + \frac{K^4}{90} = 2\tau_1 K^2 + \frac{8}{45} K^4 . \tag{4.12}$$

Here E_2 is a constant to be found as part of the solution. Substituting (4.11) into (4.4) we obtain

$$2F_{20}\tau'_{02} + \tau_1 \eta'''_{02} - \frac{1}{45}\eta^{(v)}_{02} = (2F_{21}K - \frac{3}{2} KE_2)\sin Kx$$
$$+ (4F_{21}KE_2 - \frac{3}{2} K)\sin 2Kx - \frac{9}{2} KE_2 \sin 3Kx - 3K(E_2)^2 \sin 4Kx . \tag{4.13}$$

The solutions of (4.13) are periodic and bounded if and only if the coefficients of $\sin Kx$ and $\sin 2Kx$ in the right hand side of (4.13) vanish. Thus we have

$$E_2 = \pm \frac{1}{\sqrt{2}} \tag{4.14}$$

$$F_{21} = \pm \frac{3}{4}\frac{1}{\sqrt{2}} . \tag{4.15}$$

Substituting these results into (4.1) and (4.2) we obtain

$$\eta_0 = a \cos Kx \pm a2^{-1/2} \cos 2Kx + O(a^2) , \tag{4.16}$$

$$F_2 = \frac{\tau_1 K^2}{2} + \frac{K^4}{90} \pm 3a2^{-5/2} + O(a^2) . \tag{4.17}$$

Relations (4.16) and (4.17) show that two solutions exist when (4.10) is satisfied. Similar properties were found by Wilton (1915), Pearson and Fife (1961), Schwartz and Vanden-Broeck (1979) and Chen and Saffman (1980) for waves in water of infinite depth and by Nayfeh (1970) for waves of small amplitude in water of moderate depth. It is interesting to note that the solutions (4.13), (4.14) given by Nayfeh (1970) converge to (4.16) and (4.17) as the depth tends to zero. Thus Nayfeh's solution and the present long wave calculation overlap.

A non-uniqueness analogous to the one described by (4.16) and (4.17) will occur in general when waves of wavenumber K and nK travel with the same phase velocity, i.e. when

$$\frac{\tau_1 K^2}{2} + \frac{K^4}{90} = \frac{n^2 \tau_1 K^2}{2} + \frac{n^4 K^4}{90} . \tag{4.18}$$

Here n is an arbitrary integer greater than one. The solution of (4.3) is then

$$\eta_0 = \cos Kx + E_n \cos nKx . \tag{4.19}$$

For $n = 2$, (4.18) and (4.19) reduce to (4.11) and (4.12).

Relation (4.18) can be rewritten as

$$K^2 = - \frac{45\tau_1}{1 + n^2} . \tag{4.20}$$

It follows from (3.2), (3.14) and (4.20) that this non-uniqueness can only occur for $\tau < 1/3$.

Numerical values of F_2 for $a = 0.01$ and $K = 1$ obtained by integrating (3.16) numerically are shown in Figure 1. The broken line corresponds to the solution (4.7), (4.8) and the two crosses to the solutions (4.17). These results indicate that the two solutions (4.17) are members of two different families of solutions.

Other numerical solutions of (3.16) will be discussed in Section 7.

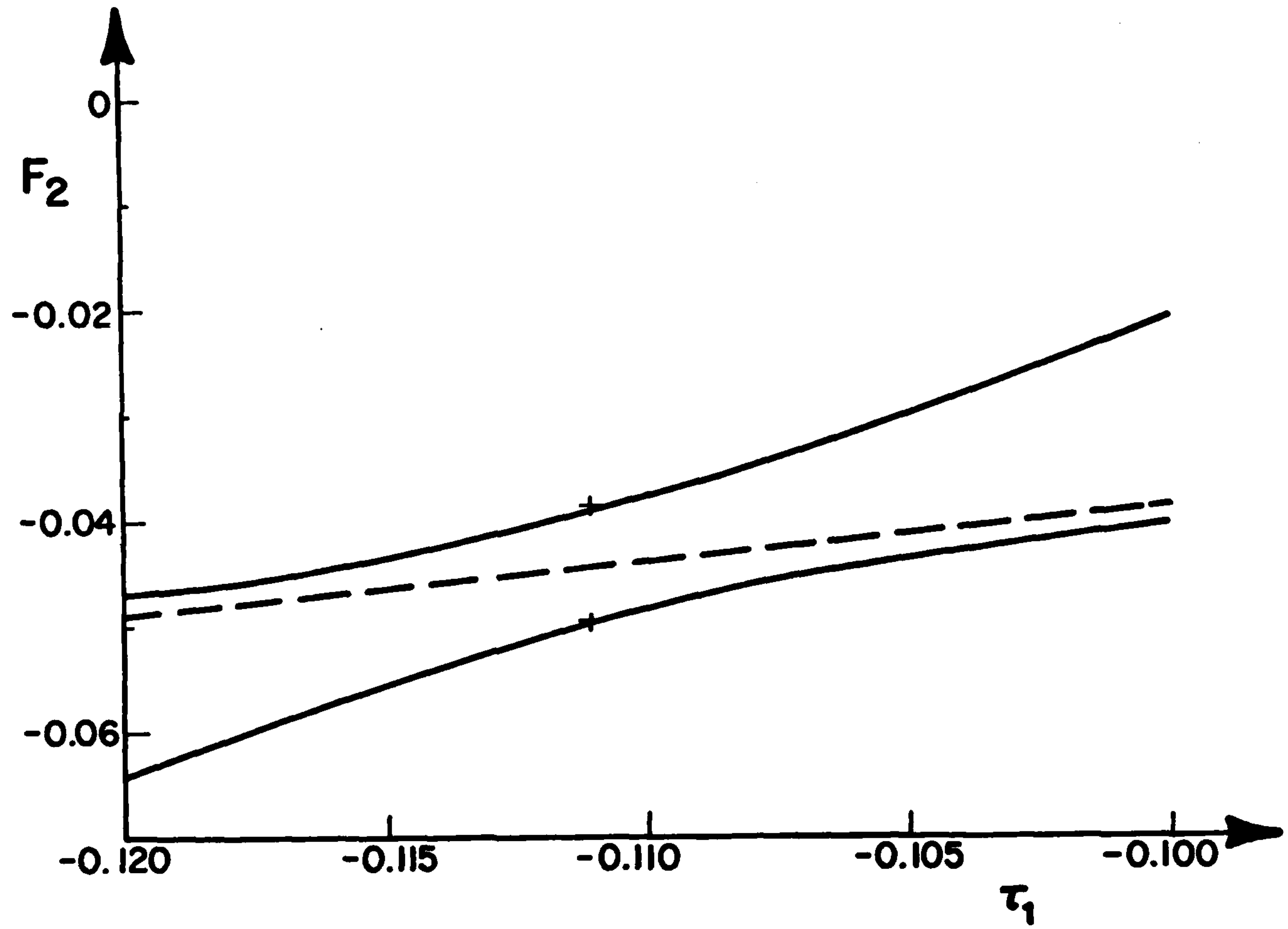


Figure 1 - Values of F_2 as a function of τ_1 for $a = 0.01$ and $K = 1$. The solid curves correspond to the numerical integration of (3.16), the broken line to the solution (4.7), (4.8) and the two crosses to the solutions (4.17).

5. Reformulation as an Integro-differential Equation

It is convenient to reformulate the problem as an integro-differential equation by considering the complex velocity $u - iv$. Here u and v are the horizontal and vertical components of the velocity respectively. The variables are made dimensionless by using H as the unit length and C as the unit velocity. We choose the complex potential

$$f = \phi + i\psi \tag{5.1}$$

as the independent variable.

In order to satisfy the boundary condition (2.8) on the bottom $\psi = -1$, we reflect the flow in the boundary $\psi = -1$. Thus we seek $x + iy$ and $u - iv$ as analytic functions of f in the strip $-2 < \psi < 0$.

Next we define the dimensionless wavelength $\ell = \lambda/H$ and introduce the change of variables

$$f = \phi + i\psi = -i \frac{\ell}{2\pi} \log p \tag{5.2}$$

where

$$p = re^{i\theta} . \tag{5.3}$$

Relations (5.2) and (5.3) map the bottom $\psi = -1$, the free surface $\psi = 0$ and its image $\psi = -2$, respectively, onto the circles $r = r_0 = e^{\frac{2\pi}{\ell}}$, $r = 1$ and $r = r_0^2$.

The values of the real and imaginary parts of the function $G(p) = u - iv - 1$ on the free surface $r = 1$ and its image $r = r_0^2$ are related by the identities

$$u(\phi) = 1 + \mathrm{Real}\{G(e^{i\theta})\} = 1 + \mathrm{Real}\{G(r_0^2 e^{i\theta})\} , \tag{5.4}$$

$$v(\phi) = -\mathrm{Im}\{G(e^{i\theta})\} = \mathrm{Im}\{G(r_0^2 e^{i\theta})\} \tag{5.5}$$

where

$$\phi = \frac{\ell}{2\pi} \theta . \tag{5.6}$$

The functions $u(\phi)$ and $v(\phi)$ in (5.4) and (5.5) are the horizontal and vertical components of the velocity on the free surface $\psi = 0$.

In order to find a relation between u and v we apply Cauchy's theorem to the function $G(p)$ in the annulus $r_0^2 < |p| < 1$. Using the relations (5.4)-(5.6) and exploiting the bilateral symmetry of the wave about $\theta = 0$ we find after some algebra

$$u(\theta) - 1 = -\frac{2}{\ell}\int_0^{\ell/2} [u(s) - 1]ds$$

$$+ \frac{1}{\ell}\int_0^{\ell/2} v(s)[\mathrm{cotg}\,\frac{\pi}{\ell}(s - \theta) + \mathrm{cotg}\,\frac{\pi}{\ell}(s + \theta)]ds$$

$$+ \frac{2}{\ell} r_0^2 \int_0^{P\ell/2} \frac{[u(s) - 1][r_0^2 - \cos\frac{2\pi}{\ell}(s - \theta)] + v(s)\sin\frac{2\pi}{\ell}(s - \theta)}{1 + r_0^4 - 2r_0^2\cos\frac{2\pi}{\ell}(s - \theta)} ds \tag{5.7}$$

$$+ \frac{2}{\ell} r_0^2 \int_0^{P\ell/2} \frac{[u(s) - 1][r_0^2 - \cos\frac{2\pi}{\ell}(s + \theta)] + v(s)\sin\frac{2\pi}{\ell}(s + \theta)}{1 + r_0^4 - 2r_0^2\cos\frac{2\pi}{\ell}(s + \theta)} ds \ .$$

The third integral in (5.7) is of Cauchy principal value form.

The surface condition (2.10) can now be rewritten as

$$\frac{1}{2}F^2[u(\phi)]^2 + \frac{1}{2}F^2[v(\phi)]^2 + \int_0^{\phi} \frac{v(s)}{[u(s)]^2 + [v(s)]^2} ds$$

$$- \tau \frac{u(\phi)v'(\phi) - v(\phi)u'(\phi)}{\{[u(\phi)]^2 + [v(\phi)]^2\}^{1/2}} = \frac{1}{2}F^2[u(0)]^2 - \frac{\tau v'(0)}{u(0)} \ . \tag{5.8}$$

In the remaining part of the paper we shall choose coordinates $\tilde{x},\tilde{y}$ with the origin at a crest or a trough of the wave. The shape of the free surface is then defined parameterically by the relations

$$\tilde{x}(\phi) = \int_0^{\phi} u(s)\{[u(s)]^2 + [v(s)]^2\}^{-1} ds \ , \tag{5.9}$$

$$\tilde{y}(\phi) = \int_0^{\phi} v(s)\{[u(s)]^2 + [v(s)]^2\}^{-1} ds \ . \tag{5.10}$$

Finally we impose the periodicity condition

$$\tilde{x}(\ell/2) = \ell/2 \ . \tag{5.11}$$

We shall measure the amplitude of the wave by the parameter

$$u_0 = u(0) \ . \tag{5.12}$$

For given values of τ, u_0 and ℓ, (5.7)-(5.12) define a system of integro-differential equations for $u(\phi)$, $v(\phi)$, $x(\phi)$, $y(\phi)$ and F.

The equations for solitary waves are obtained by taking the limit $\ell \to \infty$ in (5.7). This leads after some algebra to

$$u(\theta) - 1 = \frac{1}{\pi}\int_0^\infty v(s)\left[\frac{1}{s-\theta} + \frac{1}{s+\theta}\right]ds + \frac{1}{\pi}\int_0^\infty \frac{(s-\theta)v(s) + 2[u(s)-1]}{(s-\theta)^2+4}\,ds$$
$$+ \frac{1}{\pi}\int_0^\infty \frac{(s+\theta)v(s) + 2[u(s)-1]}{(s+\theta)^2+4}\,ds \,. \tag{5.13}$$

In the next section we describe a numerical scheme to solve these equations.

6. Numerical procedure

(a) Periodic waves

To solve the system (5.7)-(5.12) we introduce the N mesh points

$$\phi_I = \frac{\ell(I-1)}{2(N-1)}, \qquad I = 1,\ldots,N. \tag{6.1}$$

We also define the corresponding quantities

$$u_I = u(\phi_I), \qquad I = 1,\ldots,N, \tag{6.2}$$

$$v_I = v(\phi_I), \qquad I = 1,\ldots,N. \tag{6.3}$$

It follows from symmetry that $v_1 = v_N = 0$, so only $N - 2$ of the v_I are unknown. In addition from (5.12) we have $u_1 = u_0$ so only $N - 1$ of the u_I are unknown. We shall also use the $N - 1$ midpoints $\phi_{I+1/2}$ given by

$$\phi_{I+1/2} = \frac{1}{2}(\phi_I + \phi_{I+1}), \qquad I = 1,\ldots,N-1. \tag{6.6}$$

We evaluate $u_{I+1/2} = u(\phi_{I+1/2})$, $v_{I+1/2} = v(\phi_{I+1/2})$, $u'_{I+1/2} = u'(\phi_{I+1/2})$, and $v'_{I+1/2} = v'(\phi_{I+1/2})$ by four points interpolation and difference formulas.

We now discretize (5.7) by applying the trapezoidal rule to the integrals on the right-hand side with $s = \phi_I$, $I = 1,\ldots,N$, and $\theta = \phi_{I+1/2}$, $I = 1, N-1$. The symmetry of the quadrature formula and of the discretization enables us to evaluate the Cauchy principal value integral as if it were an ordinary integral. In this way we obtain $N - 1$ algebraic equations.

Next we substitute into (5.8) the expressions for $u_{I+1/2}$, $v_{I+1/2}$, $u'_{I+1/2}$ and $v'_{I+1/2}$ at the points $\phi_{I+1/2}$, $I = 2,\ldots,N-1$. The integral in (5.8) is evaluated by the trapezoidal rule with the mesh points $s = \phi_I$. The derivative $v'(0)$ in (5.8) is approximated by a four points difference formula. Thus we obtain another $N - 2$ algebraic equations.

For given values of τ, u_0 and ℓ we have therefore $2N - 3$ algebraic equations for the $2N - 2$ unknowns u_I, v_I and F. The last equation is obtained by discretizing (5.11).

The $2N - 2$ equations are solved by Newton's method. After a solution converges for given values of τ, u_0 and ℓ, the surface profile $\tilde{x}(\phi)$, $\tilde{y}(\phi)$ is obtained by applying the trapezoidal rule to (5.9) and (5.10).

For each calculation presented in Section 7 the number of mesh points N was progressively increased up to a value for which the results were independent of N within graphical accuracy. Most of the computations were performed with $N = 60$. A few runs were performed with $N = 100$ as a check on the accuracy of the computations.

In the remaining part of the paper we shall refer to the above numerical scheme for periodic waves as numerical scheme I.

(b) Solitary waves

To solve the system (5.8)-(5.13) we introduce the N mesh points

$$\phi_I = E(I - 1), \qquad I = 1,\ldots,N . \tag{6.5}$$

Here E is the interval of discretization.

The quantities u_I, v_I, $\phi_{I+1/2}$, $u_{I+1/2}$ and $v_{I+1/2}$ are defined as in the previous subsection. It follows from symmetry that $v_1 = 0$. In addition $u_1 = u_0$ so that only $2N - 2$ of the u_I and v_I are unknown.

The discretization of (5.13) is entirely analogous to the procedure used to discretize (5.7). In this way we obtain $N - 1$ algebraic equations. The truncation error due to approximating the infinite integrals by integrals over a finite range, was found to be negligible for NE sufficiently large. As shown in the previous subsection, the discretization of (5.8) leads to another $N - 2$ algebraic equations.

Thus, for given values of τ, u_0 and ℓ we have $2N - 3$ algebraic equations for the $2N - 1$ unknowns u_I, v_I and F. The last two equations are obtained by imposing $u_N = 1$ and $v_N = 0$.

We shall refer to this numerical scheme as numerical scheme II.

7. Discussion of the Results

(a) Depression solitary waves

Numerical scheme II was used to compute depression solitary waves for given values of τ and u_0. In the first calculation, the Newton iterations were started with the asymptotic solution (2.19) as the initial guess. For u_0 close to one, the iterations converged rapidly. Once a solution was obtained it was used as the initial guess for the next calculation with a slightly different value of τ or u_0. The curve a in Figure 2 shows a typical profile for $\tau = 0.7$ and $u_0 = 3.0$.

The asymptotic solution (2.19) can be rewritten in terms of the variables used in the numerical scheme as

$$\tilde{y} = A\left[\operatorname{sech}^2\left[\frac{3A}{4(1-3\tau)}\right]^{1/2}\tilde{x} - 1\right] , \tag{7.1}$$

$$F^2 = 1 + A . \tag{7.2}$$

The curve b in Figure 2 corresponds to the profile (7.1) in which the amplitude A is equal to the amplitude of the numerical solution. For $1 < u_0 < 1.03$ the numerical results and the asymptotic formula (7.1) were found to be indistinguishable to graphical accuracy.

In Table 1 we compare numerical values of F with the approximation (7.2) for various values of u_0 and $\tau = 0.4$, and 0.7.

As u_0 increases for a given value of $\tau > \frac{1}{3}$, the wave profile becomes steeper and the distance between the trough and the bottom decreases. For $\tau > \frac{1}{2}$ this distance tends to zero as $u_0 \to \infty$. The corresponding Froude number tends to zero and the profile approaches a static limiting configuration in which gravity is balanced by surface tension. Then (2.10) reduces to a differential equation for the free surface:

$$\eta - \tau\eta_{xx}(1 + \eta_x^2)^{-3/2} = 1 . \tag{7.3}$$

The boundary conditions for (7.3) are

$$\eta(0) = 0 \tag{7.4}$$

$$\eta(\infty) = 1 . \tag{7.5}$$

TABLE 1

u_0	$\tau = 0.4$ numerical	$\tau = 0.4$ KdV	$\tau = 0.7$ numerical	$\tau = 0.7$ KdV
1.0	1.0	1.0	1.0	1.0
1.03	0.986	0.986	0.986	0.985
1.5	0.844	0.850	0.832	0.831
2.0	0.756	0.764	0.734	0.734
4.0	0.578	0.590	0.541	0.542
10.0	0.396	0.407	0.355	0.355
20.0	0.291	0.301	0.255	0.253
50.0	0.190	0.195	0.163	0.152
100.0	0.136	0.137	0.116	0.081

Table 1: Values of the Froude number for depression solitary waves for $\tau = 0.4$ and $\tau = 0.7$, and various values of u_0. Numerical values were computed by scheme II, and the KdV values found from (7.2) with A taken equal to the amplitude of the numerical solution.

Multiplying (7.3) by η_x and integrating with respect to x yields

$$\frac{\eta^2}{2} + \tau\left(1 + \eta_x^2\right)^{-1/2} = \eta + \tau - \frac{1}{2} . \tag{7.6}$$

The value of the constant of integration in (7.6) was evaluated by using (7.5). Integrating (7.6) gives a formula for the shape of the free surface, namely

$$x = \int_0^{\eta} \left[\tau^2\left(\eta + \tau - \frac{1}{2} - \frac{\eta^2}{2}\right)^{-2} - 1\right]^{-1/2} d\eta . \tag{7.7}$$

The curve c in Figure 2 represents the profile (7.7) for $\tau = 0.7$.

We denote by $\tan \sigma$ the slope of the profile (7.7) at the trough $x = 0$. Then (7.4) and (7.6) yield

$$\cos \sigma = 1 - \frac{1}{2\tau} . \tag{7.8}$$

Relation (7.8) implies that the present limiting configuration is only possible for $\tau > 1/2$.

For $\tau < 1/2$, the numerical computations indicate that the wave ultimately reaches a critical configuration with a trapped bubble at the trough. This critical configuration is shown in Figure 3 for $\tau = 0.4$. Similar limiting configurations were obtained previously by Crapper (1957), Schwartz and Vanden-Broeck (1979) and Vanden-Broeck and Keller (1980). Waves for larger values of u_0 could be obtained by allowiong the pressure in the trapped bubble to be different from the atmospheric pressure (Vanden-Broeck and Keller (1980)).

Numerical scheme II was used to compute depression solitary waves for $\tau < 1/3$. In Figure 4 we present solutions for $u_0 = 1.03$ and various values of τ. As τ decreases the profiles develop a large number of inflexion points. We were unable to compute solutions for $u_0 = 1.03$ and $\tau < 0.21$ because too many mesh points were required.

In Figure 5 we compare the numerical solution of the exact nonlinear equations with the profile obtained by numerically integrating (3.16). We found that the two solutions become identical within graphical accuracy in the limit as $\tau \to 1/3$ and $u_0 \to 1$ with the ratio $(u_0 - 1)(\tau - 1/3)^{-2}$ constant. This constitutes an important check on the consistency of our results.

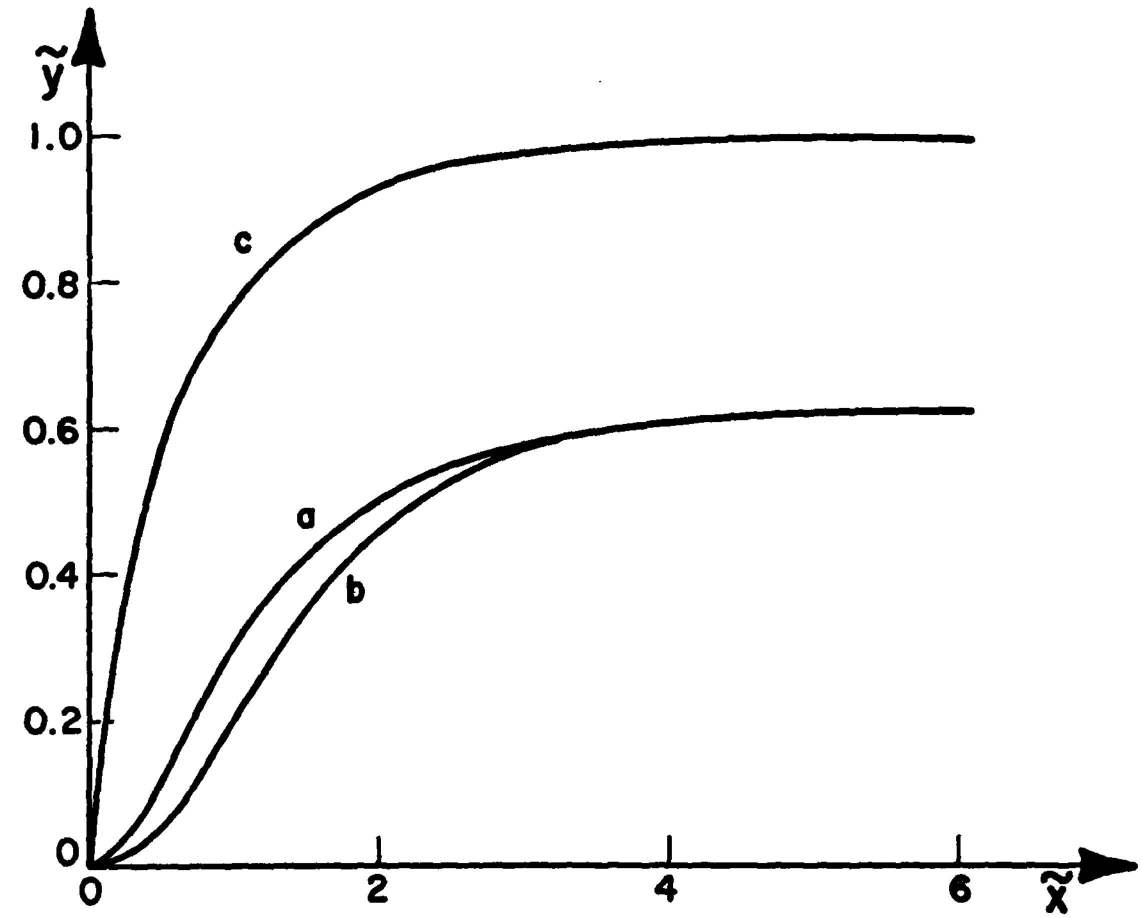


Figure 2 - Free surface profiles of depression solitary waves for $\tau = 0.7$. The curve a is the exact numerical solution for $u_0 = 3.0$. The curve b corresponds to the KdV approximation (7.1) in which A is equal to the amplitude of curve a. The curve c corresponds to the limiting profile (7.1).

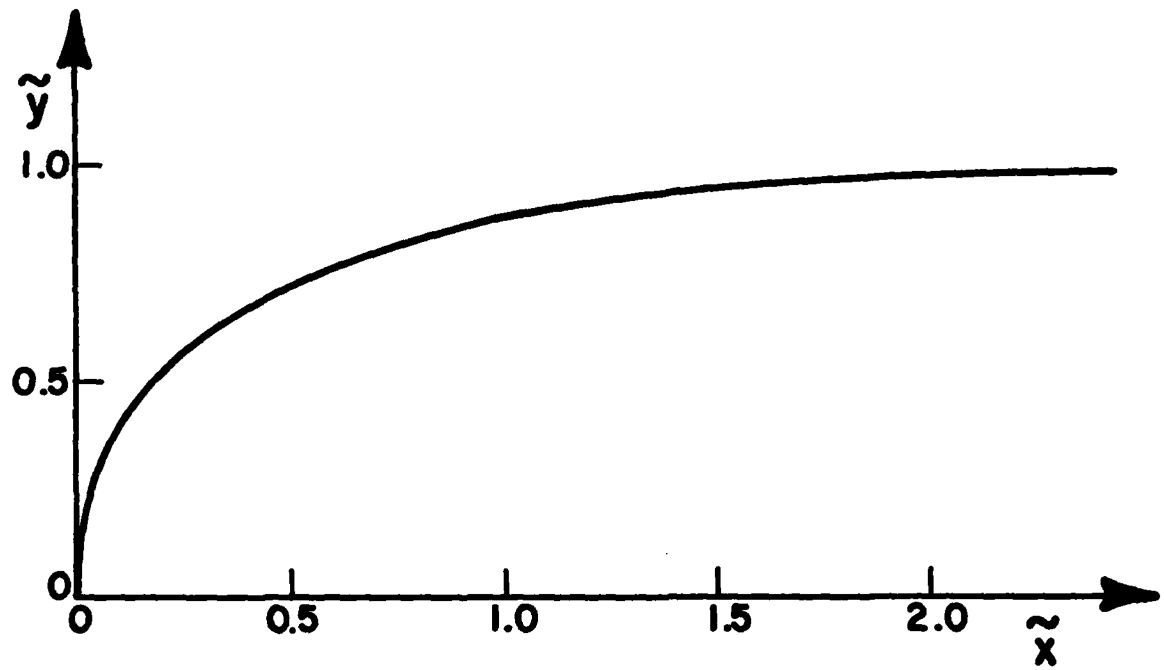


Figure 3a - Computed free surface profile of a depression solitary wave for $\tau = 0.4$ and $u_0 = 227$. The free surface has just one point of contact with itself and encloses a small bubble at the trough.

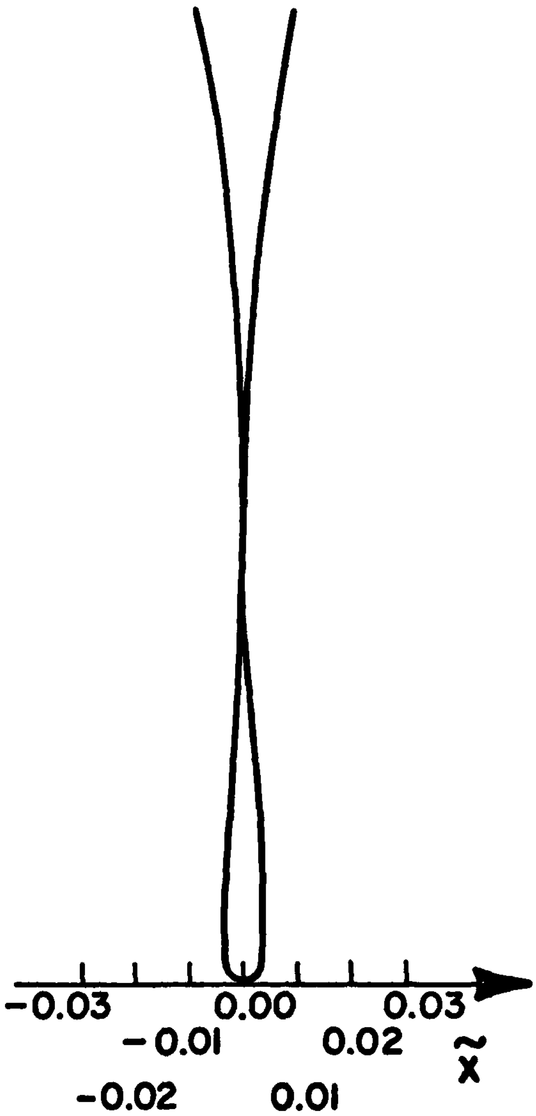


Figure 3b - The bubble of Figure 3a expanded by a factor of 12.5. The vertical scale is the same as the horizontal scale.

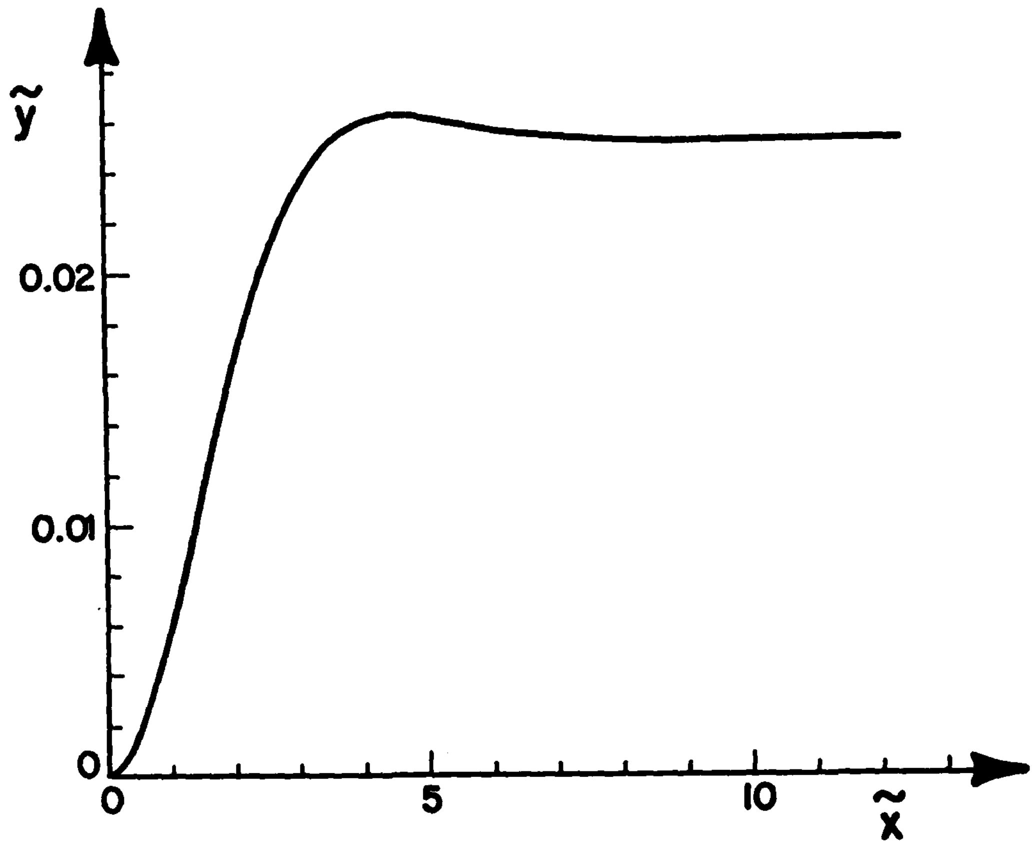


Figure 4a - Computed free surface profile of a depression solitary wave for $u_0 = 1.03$ and $\tau = 0.32$. The Froude number if $F = 0.985$.

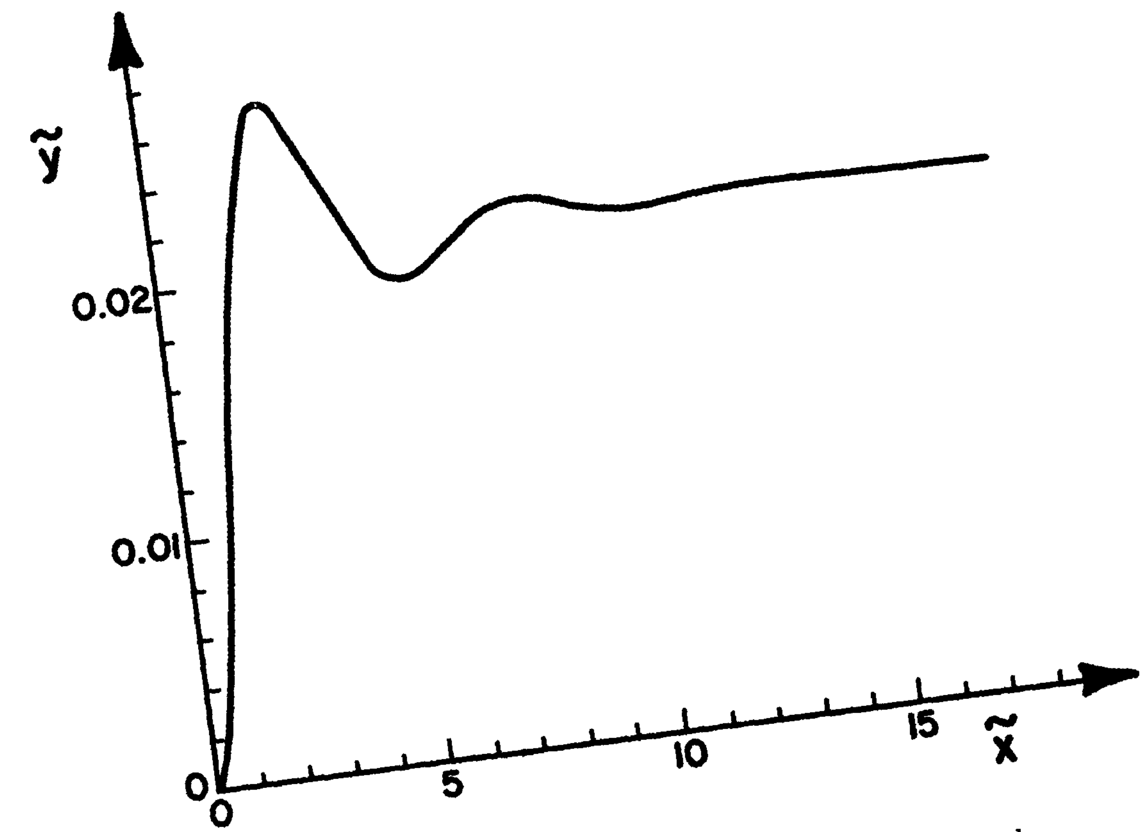


Figure 4b - Same as Figure 4a with $u_0 = 1.03$, $\tau = 0.27$ and $r = 0.977$.

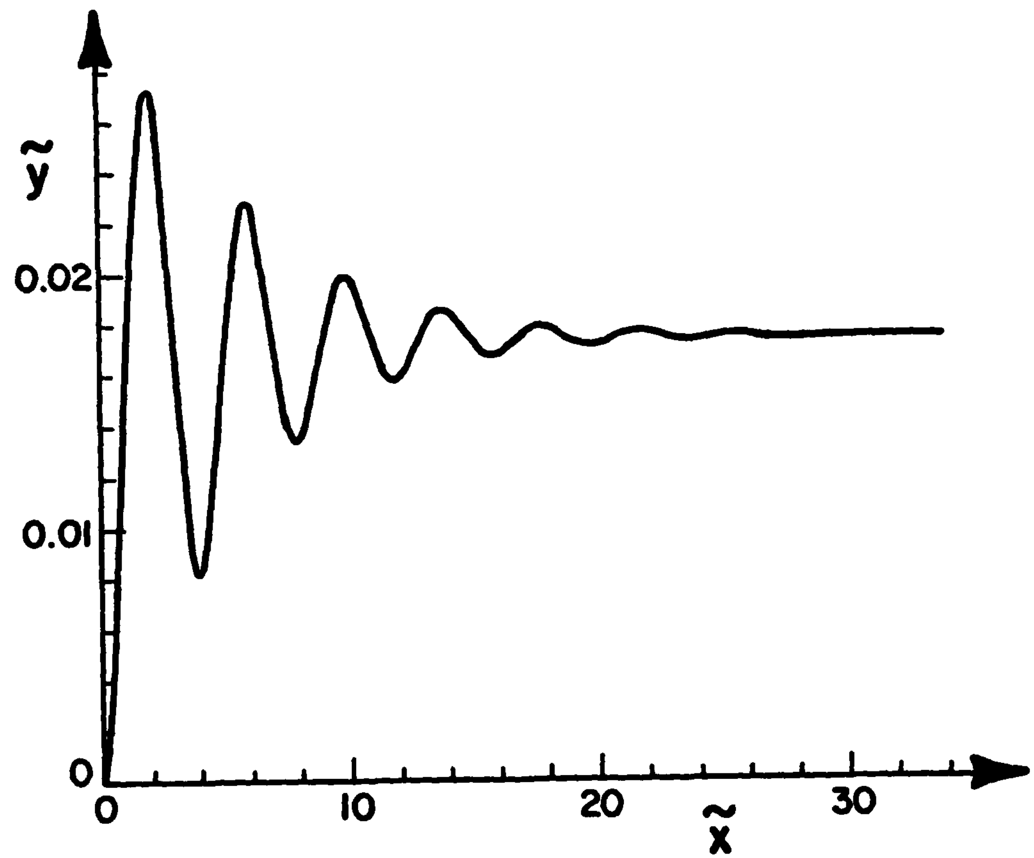


Figure 4c - Same as Figure 4a with $u_0 = 1.03$, $\tau = 0.228$ and $F = 0.954$.

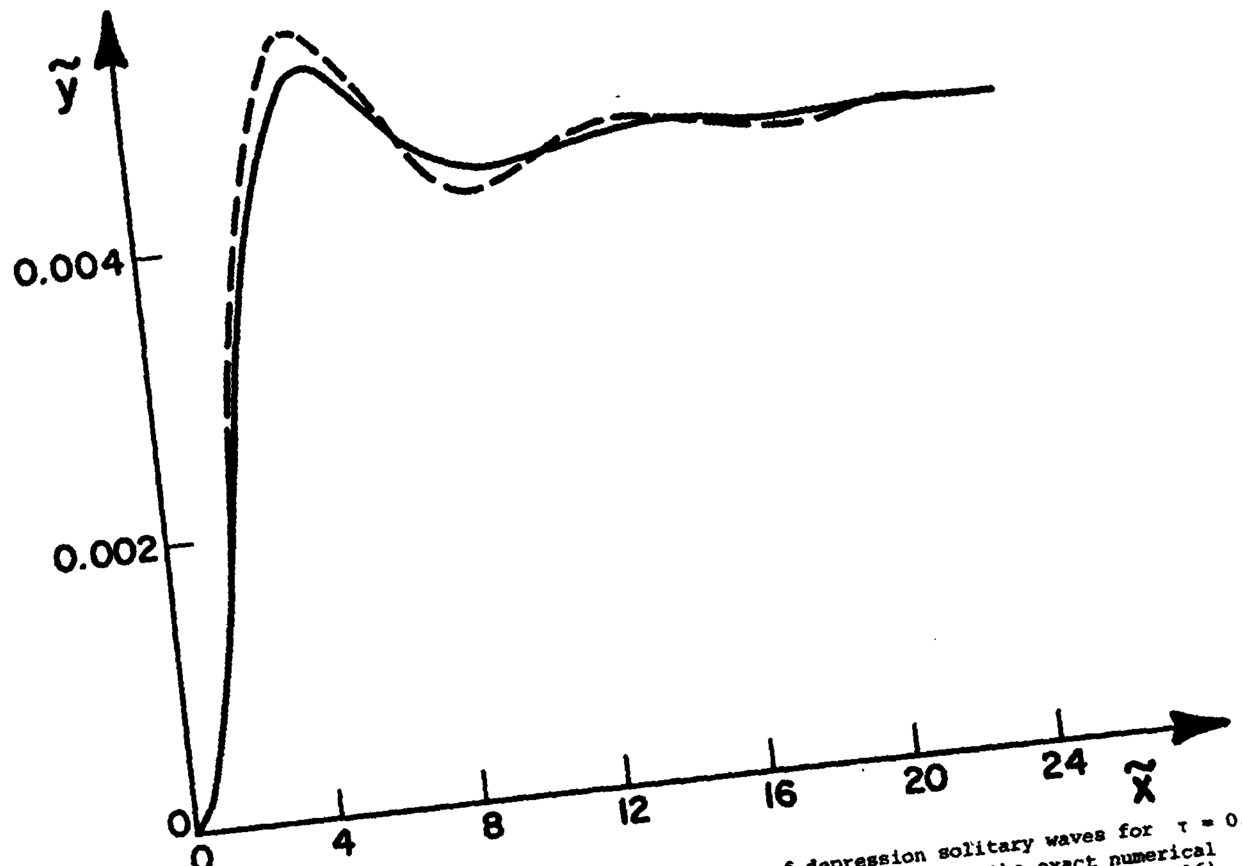


Figure 5 - Computed free surface profiles of depression solitary waves for $\tau = 0.316$ and $u_0 = 1.005$. The solid curve corresponds to the exact numerical solution and the broken curve to the numerical integration of (3.16). The value of the Froude number is $F = 0.997$.

(b) Elevation solitary waves

In preliminary computations we attempted to calculate solitary waves for $\tau < 1/3$ by using numerical scheme II. The iterations converged rapidly. However, the profiles were not independent of N and E and they did not appear to approach a limiting profile as $NE \to \infty$ and $E \to 0$. Furthermore at large x the profiles oscillated with finite amplitude about the undisturbed level instead of approaching a uniform stream.

On the other hand we were able to compute periodic waves of large wavelength ℓ *for* $\tau < 1/3$, by using numerical scheme I. A large number of dimples are present on these profiles (see Figure 6). We found many different families of periodic waves. This non-uniqueness agrees with the perturbation calculation of Section 4. Waves in different families are characterized by the number of dimples in a wavelength. In Figure 7 we plot the Froude number versus wavelength for 3 such families, when $\tau = 0.24$. Curve a corresponds to 14 dimples, curve b to 15 dimples, and curve c to 16 dimples. Graphs of representive members of each of these families are presented in Figure 6.

As can be seen from Figure 7 each family only exists for a limited range of wavelengths. In particular there is a maximum value of the wavelength beyond which solutions in a given family cease to exist. Therefore, a solitary wave cannot be obtained as a limit of solutions belonging to a single family.

In conclusion all our attempts to find elevation solitary with surface tension by either numerical scheme I or II failed. Mathematically the question of the existence of these elevation waves remain open.

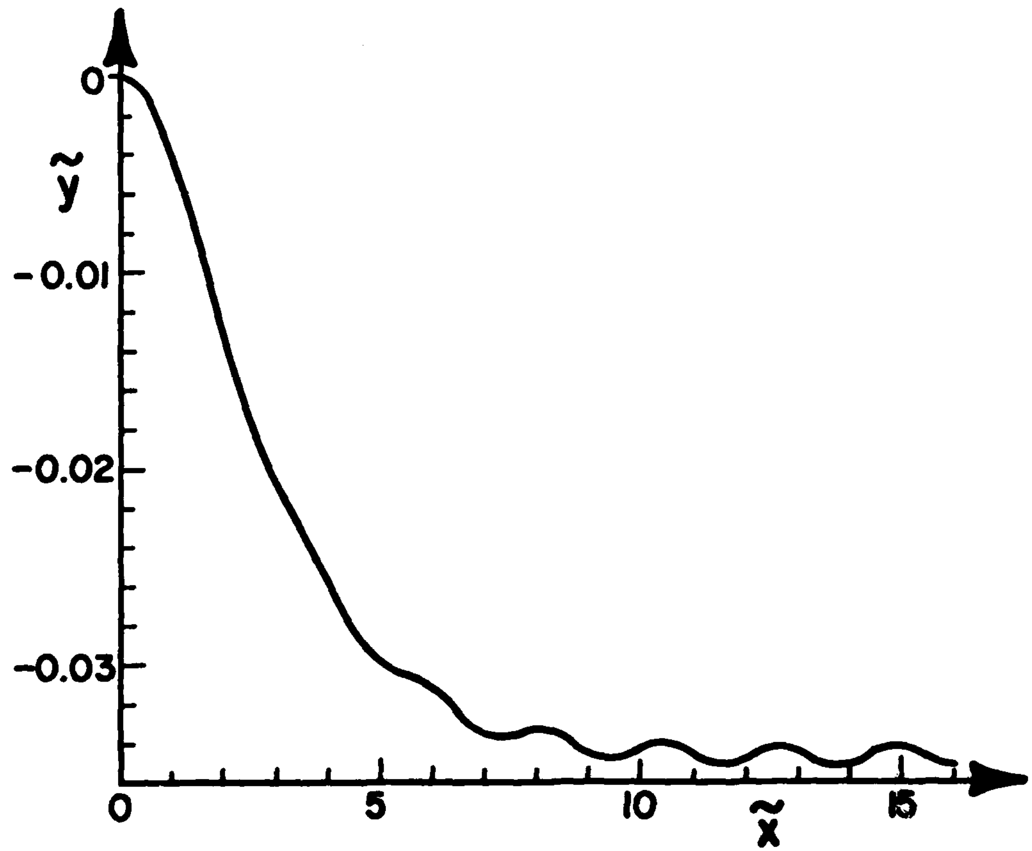


Figure 6a - Computed free surface profile of a periodic wave with $\tau = 0.24$ and $u_0 = 0.97$. The wavelength is $\ell = 32$.

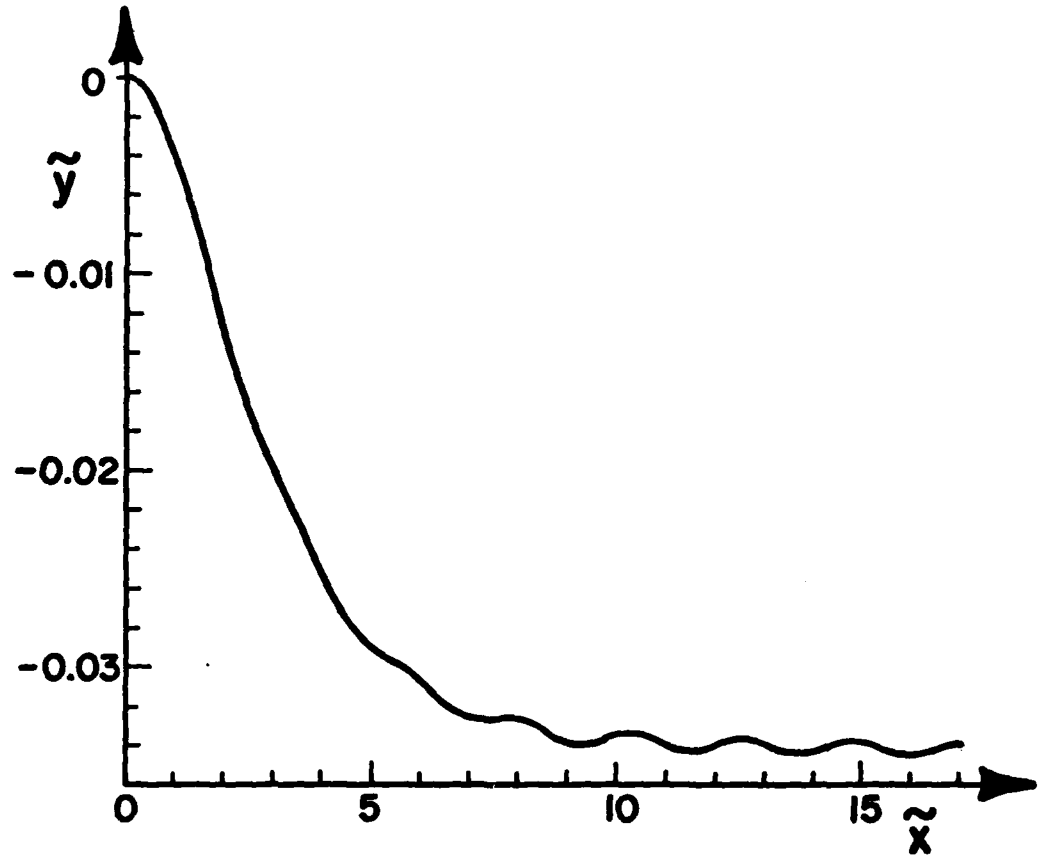


Figure 6b - Same as in Figure 6a with $\ell = 34$.

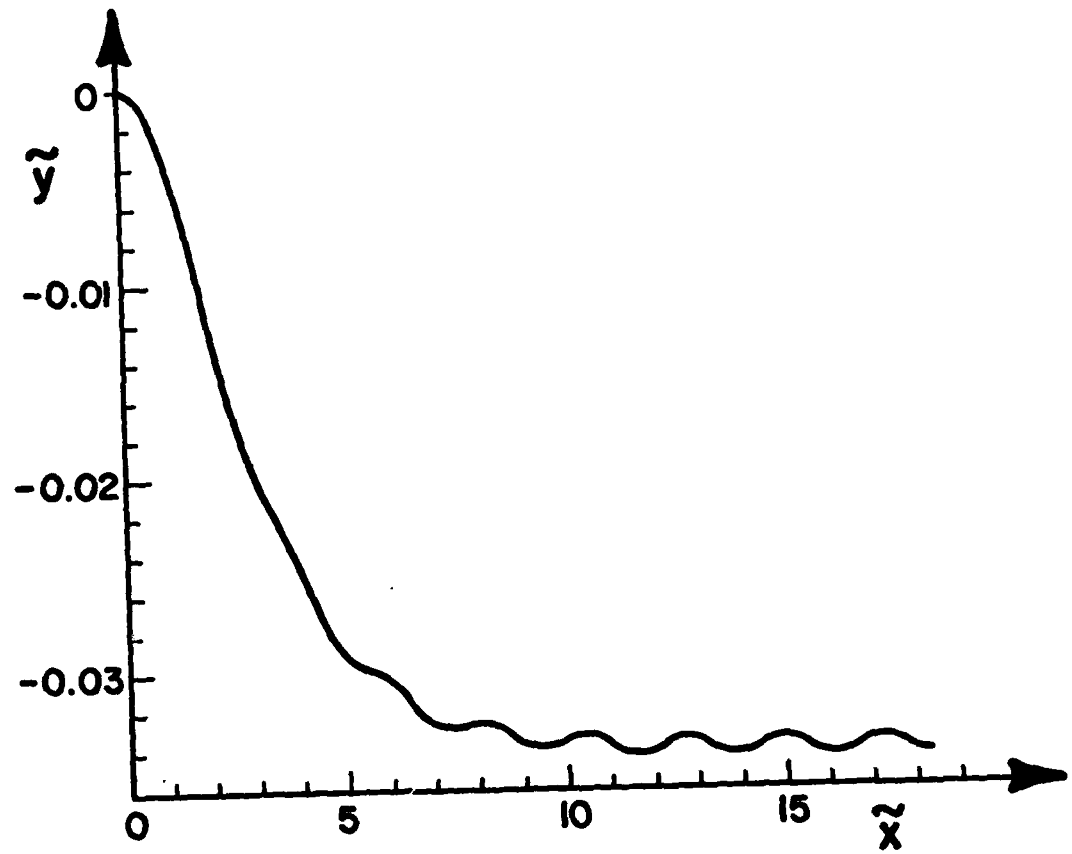


Figure 6c - Same as in Figure 6a with $\ell = 36.6$.

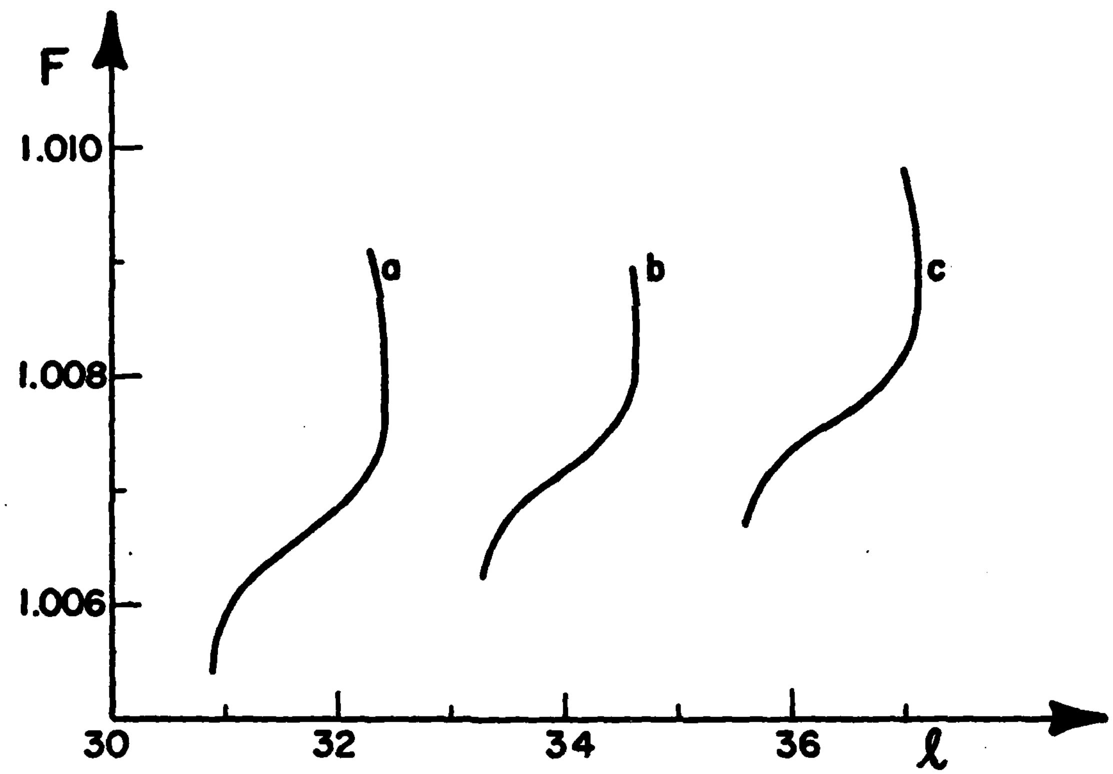


Fig.7 - Computed values of the Froude number F versus the wavelength ℓ for periodic waves with $\tau = 0.24$ and $u_0 = 0.97$. Curve a corresponds to 14 dimples, per wavelength, curve b to 15 dimples and curve c to 16 dimples.

REFERENCES

Benjamin, T. B. 1982 Quart. Appl. Math. 37, 183.

Byatt-Smith, J. G. B. and Longuet-Higgins, M. S. 1976 Proc. Roy. Soc. A350, 175.

Chen, B. and Saffman, P. G. 1980 Studies in Appl. Math. 62, 95.

Cokelet, E. D. 1977 Phil. Trans. Roy. Soc. A286, 183.

Crapper, G. D. 1957 J. Fluid Mech. 2, 532.

Harrison, W. J. 1909 Proc. Lond. Math. Soc. 7, 107.

Hunter, J. K. and Vanden-Broeck, J.-M. 1982 (to appear).

Keller, J. B. 1948 Comm. Pure Appl. Math. 1, 323.

Korteweg, D. J. and de Vries, G. 1895 Phil. Mag. 39, 422.

Laitone, E. V. 1960 J. Fluid Mech. 9, 430.

Longuet-Higgins, M. S. and Fenton, J. D. 1974 Proc. Roy. Soc. A340, 471.

Miles, J. W. 1980 Ann. Rev. Fluid Mech. 12, 11.

Nayfeh, A. H. 1970 J. Fluid Mech. 40, 671.

Pierson, W. J. and Fife, P. 1961 J. Geophys. Res. 66, 163.

Rayleigh, Lord 1876 Phil. Mag. 1, 257.

Rienecker, M. M. and Fenton, J. D. 1981 J. Fluid Mech. 104, 119.

Schwartz, L. W. 1974 J. Fluid Mech. 62, 553.

Schwartz, L. W. and Fenton, J. D. 1982 Ann. Rev. Fluid Mech. 14, 39.

Schwartz, L. W. and Vanden-Broeck, J.-M. 1979 J. Fluid Mech. 95, 119.

Shinbrot, M. 1981 Quart. Appl. Math. 39, 287.

Stokes, G. G. 1847 Trans. Camb. Phil. Soc. 8, 441.

Stokes, G. G. 1880 Mathematical and Physical Papers, vol. 1, p. 314, Cambridge University Press.

Vanden-Broeck, J.-M. and Keller, J. B. 1980 J. Fluid Mech. 98, 161.

Vanden-Broeck, J.-M. and Shen, M. C. 1982 (to appear).

Vanden-Broeck, J.-M. and Schwartz, L. W. 1979 Phys. Fluids 22, 1868.

Whitham, G. B. 1974 Linear and Nonlinear Waves, New York, Wiley.

Wilton, J. R. 1915 Phil. Mag. 29, 688.

Witting, J. 1975 SIAM J. Appl. Math. 28, 700.

JKH:JMVB:scr

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 2414	2. GOVT ACCESSION NO.: AD-A120984
3. RECIPIENT'S CATALOG NUMBER:	
4. TITLE (and Subtitle): SOLITARY AND PERIODIC GRAVITY-CAPILLARY WAVES OF FINITE AMPLITUDE	5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): J. K. Hunter and J.-M. Vanden-Broeck	8. CONTRACT OR GRANT NUMBER(s): MCS-8001960, DAAG29-80-C-0041, MCS-7927062, Mod. 1
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: Work Unit Number 2 (Physical Mathematics)
11. CONTROLLING OFFICE NAME AND ADDRESS: See Item 18 below.	12. REPORT DATE: August 1982
	13. NUMBER OF PAGES: 33
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Solitary waves
Periodic waves
Surface tension

(4 sourced)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Two dimensional solitary and periodic waves in water of finite depth are considered. The waves propagate under the combined influence of gravity and surface tension. The flow, the surface profile, and the phase velocity are functions of the amplitude of the wave and the parameters $\ell = \lambda/H$ (lambda) and $\tau = T/\rho g H^2$. Here λ is the wavelength, H the depth, T the surface

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. ABSTRACT - cont'd.

Rho   tau

tension, ρ the density and g the gravity. For small values of τ and small values of the amplitude, the profile of the wave satisfies the Korteweg de Vries equation approximately. However, for τ close to 1/3 this equation becomes invalid. In the present paper a new equation valid for τ close to 1/3 is obtained. Moreover, a numerical scheme based on an integro-differential equation formulation is derived to solve the problem in the fully nonlinear case. Accurate solutions for periodic and solitary waves are presented. In addition, the limiting configuration for large amplitude solitary waves when $\tau > \frac{1}{2}$ is found analytically. Graphs of the results are included.

ME

8